Soundness for Linear Logic regarding Phase Semantics

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One kind of soundness for Linear Logic as to Phase Semantics is dealt with.

What is Linear Logic?

Features:

- Decomposition of Classical Logic
- Realizing constructivity keeping Duality (symmetry) intact
- Resource sensitive (i.e. each hypothesize can be used exactly at once)
- Suitable for expressing parallel computing

Decomposition of Classical Logic

Familiar connectives "$\wedge$" and "$\vee$" break into weaker four connectives:

- "$\wedge$" into "$\otimes$" and "&"
- "$\vee$" into "$\parr$" and "$\oplus$"

Another viewpoint:

- *Multiplicative* are "$\otimes$" and "$\parr$"
- *Additive* are "&" and "$\oplus$"

System containing only Multiplicative and Additive is called *MALL*

Resource sensitiveness

"$\multimap$" is linear version of "$\Rightarrow$"

Usual logic :

$$\frac{X \qquad X \Rightarrow Y}{Y \text{ (but } X \text{ still holds.)}}$$

Linear Logic :

$$\frac{X \qquad X \multimap Y}{Y \text{ (and then } X \text{ disappears!)}}$$

$\otimes$: Tensor, suitable for expressing parallel computing

$$\frac{f : A \multimap C \qquad g : B \multimap D}{f \otimes g : A \otimes B \multimap C \otimes D}$$

Meaning : Programs which do NOT share the same resouce can be executed at the same time.

& : Cartesian product

$$\frac{f : X \multimap Y \qquad g : X \multimap Z}{f \,\&\, g : X \multimap Y \,\&\, Z}$$

and

$$Y \,\&\, Z \nvdash Y \otimes Z$$

Meaning : Programs which share the same resouce CANNOT be executed at the same time unless copying the resource, while we CAN CHOOSE either Y or Z.
In fact, copying and deleting of resources are explicit.

$\oplus$: Additive Sum, dual of “&”

$$Y \oplus Z \nvdash Y$$

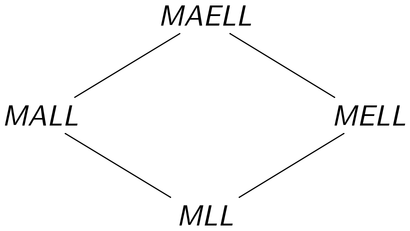
nor

$$Y \oplus Z \nvdash Z$$

Meaning : Either Y or Z holds, but we CANNOT choose neither Y or Z.

Off topic: Variants of Linear Logic

- Multiplicative are "$\otimes$" and "⅋"
- Additive are "&" and "$\oplus$"
- *Exponential* are "!" and "?"



What is Phase Semantics?

First off, what is *semantics*?

- To provide rigorous defnitions that abstract away from implementation details
- To provide mathematical tools for proving properties of programs

(Amadio, Curien)

Phase Semantics is a kind of semantics, which is based on the idea of *Tarskian style*:

- "A" means A which is truth value (true or false)
- "$A \wedge B$" means "A" and "B"

and so on.
This seems ovious, however, there is another semantics which is not the case: *Coherent Semantics*, which is BHK style inconsistent semantics.
Phase Space is Phase Semantics for MALL.

Other semantics

- Coherent semantics
- Categorical semantics
- Geometry of interaction
- Game semantics

  and so on.

What is *soundness*?

Formulae derived using specific rules are semantically valid, which is minimum requirement for semantics in general (Systems which yields lie are compeletely useless).

Syntax of MALL

Definition 1

Formula of MALL.

$$
\begin{aligned}
A ::= {} & p \mid p^{\perp} \\
& \mid A \otimes A \mid A \oplus A \\
& \mid A \mathbin{\&} A \mid A \parr A \\
& \mid \mathbf{1} \mid \mathbf{0} \mid \top \mid \bot
\end{aligned}
$$

Inference rules of MALL I

Inference rules of MALL.

$$\frac{}{\vdash A, A^{\perp}}\,(\mathit{identity}) \qquad \frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\,(\mathit{cut})$$

$$\frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}\,(\mathit{exchange})$$

$$\frac{}{\vdash \mathbf{1}}\,(\mathit{one}) \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\,(\mathit{false})$$

Inference rules of MALL II

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, A \otimes B, \Delta}\,(\mathit{times}) \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,(\mathit{par})$$

$$\frac{}{\vdash \Gamma, \top}\,(\mathit{true}) \qquad (\mathit{no\ rule\ for\ zero})$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\,(\mathit{with}) \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\,(\mathit{left\ plus})$$

$$\frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\,(\mathit{right\ plus})$$

Phase Space

We give semantics for MALL.

Definition 2

Phase Space $:= (M, \perp)$
where M is commutative monoid and $\perp \subseteq M$ is defined.

Definition 3

Commutative monoid M holds

- commutativity: $pq = qp$
- associativity: $(pq)r = p(qr)$
- identity: $1p = p1 = p$

for all $p, q, r \in M$.

Definition 4

$X \multimap Y$ is defined as

$$
\begin{aligned}
& m \in X \multimap Y \\
& :\Leftrightarrow \forall x (x \in X \Rightarrow mx \in Y)
\end{aligned}
$$

Definition 5

orthogonal

$$X^{\perp} := X \multimap \perp$$

Definition 6

X is *fact* iff

$$X = X^{\perp\perp}$$

or equivalently, X is of the form $Y^{\perp}$.

Definition 7

For convention,

$$xy \in X.Y :\Leftrightarrow x \in X \land y \in Y$$

Let X and Y be fact. Connectives are interpreted in this way (more precisely, we are defining interpretation function from *formula* to $\wp(M)$):

$$X \otimes Y := (X.Y)^{\perp\perp}$$
$$X \parr Y := (X^{\perp}.Y^{\perp})^{\perp}$$
$$X \multimap Y = (X.Y^{\perp})^{\perp}$$
$$X \mathbin{\&} Y := X \cap Y$$
$$X \oplus Y := (X \cup Y)^{\perp\perp}$$
$$\mathbf{1} := \{1\}^{\perp\perp}$$
$$\mathbf{0} := \varnothing^{\perp\perp}$$
$$\top := M$$

Validity

Definition 8

Sequent

$$\vdash \Gamma, A$$

is interpreted as subset of M

$$\Gamma \parr A$$

Definition 9

$\underline{X}$ (as formula) is valid iff $1 \in \underline{X}$ ($\underline{X} \subseteq M$)

Proof of soundness I

Theorem 10

Sequent which are provable in MALL are all valid in Phase Space i. e.

$$\vdash \underline{X} \Rightarrow 1 \in \underline{X}$$

where $\underline{X}$ are all fact.

By straightforward induction on inference rules.

1. $\dfrac{}{\vdash A, A^{\perp}}\ (identity)$

$$A \parr A^{\perp} = A \multimap A \ni 1$$

($\because$ definition of identity 1)

Proof of soundness II

2. $\dfrac{}{\vdash \Gamma, \top}\,(\mathit{true})$

$$\Gamma \parr \top = \mathbf{0} \multimap \Gamma$$

Since $\mathbf{0}$ is the smallest fact, $\mathbf{0} \subseteq \Gamma$. This implies $\forall z, z \in \mathbf{0} \Rightarrow z \in \Gamma$.
Hence $\forall z, z \in \mathbf{0} \Rightarrow 1z \in \Gamma$.
$\therefore$ By definition of "$\multimap$", $1 \in \mathbf{0} \multimap \Gamma = \Gamma \parr \top$

3. $\dfrac{}{\vdash \mathbf{1}}\,(\mathit{one})$
Oviously,

$$1 \in \{1\} \subseteq \{1\}^{\perp\perp} = \mathbf{1}$$

Proof of soundness III

4. $$\frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ (cut)$$
   We have to show that if $1 \in \Gamma \parr A$ and $1 \in A^{\perp} \parr \Delta$ then $1 \in \Gamma \parr \Delta$. In fact this is equivalent to
   $$\begin{aligned} &1 \in \Gamma^{\perp} \multimap A, 1 \in A \multimap \Delta \\ \Rightarrow &1 \in \Gamma^{\perp} \multimap \Delta \end{aligned}$$
   This is easily followed by
   $$\begin{aligned} &\Gamma^{\perp} \subseteq A, A \subseteq \Delta \\ \Rightarrow &\Gamma^{\perp} \subseteq \Delta \end{aligned}$$
5. $$\frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}\ (exchange)$$
   $\because$ "$\wedge$" is commutative.

Proof of soundness IV

6. $$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\ (\mathit{with})$$
We have to show that if $1 \in \Gamma \parr A$ and $1 \in \Gamma \parr B$ then $1 \in \Gamma \parr (A \,\&\, B)$. Here we use distributivity of $\parr$ over &:

$$(\Gamma \parr A) \,\&\, (\Gamma \parr B) = \Gamma \parr (A \,\&\, B)$$

By definition of & and hypothesis, left hand side contains 1, and so does right hand side.

7. $$\frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ (\mathit{left\ plus}) \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ (\mathit{right\ plus})$$
Similarly, we use half-distributivity of $\parr$ over $\oplus$:

$$(\Gamma \parr A) \oplus (\Gamma \parr B) \subseteq \Gamma \parr (A \oplus B)$$

Proof of soundness V

8. $\dfrac{\vdash \Gamma}{\vdash \Gamma, \bot}$ (*false*)

$$\Gamma \parr \bot = \mathbf{1} \multimap \Gamma$$

By definition, $\mathbf{1}$ is the smallest fact contains 1 i. e. $\mathbf{1} \subseteq \Gamma$ so that $\forall e, e \in \mathbf{1} \Rightarrow e = 1e \in \Gamma$ hence $1 \in \mathbf{1} \multimap \Gamma$.

9. $\dfrac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, A \otimes B, \Delta}$ (*times*)
We have to show that if $1 \in \Gamma \parr A$ and $1 \in \Delta \parr B$ then $1 \in \Gamma \parr \Delta \parr (A \otimes B)$. Hypotheses can be transformed into

Proof of soundness VI

$1 \in \Gamma^{\perp} \multimap A$ and $1 \in \Delta^{\perp} \multimap B$ respectively. Therefore, we have $\Gamma^{\perp} \subseteq A$ and $\Delta^{\perp} \subseteq B$

$$
\begin{aligned}
&\Rightarrow \Gamma^{\perp}.\Delta^{\perp} \subseteq A.B \\
&\Rightarrow (\Gamma^{\perp}.\Delta^{\perp})^{\perp\perp} \subseteq (A.B)^{\perp\perp} \\
&\Rightarrow \Gamma^{\perp} \otimes \Delta^{\perp} \subseteq A \otimes B \\
&\Rightarrow \Gamma^{\perp} \otimes \Delta^{\perp} \multimap A \otimes B \ni 1 \\
&\Rightarrow \Gamma \parr \Delta \parr (A \otimes B) \ni 1
\end{aligned}
$$

10. $\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,(par)$
This is tautology. $\square$

Future work

- Categorical semantics: symmetrical monoidal (closed) category
- Application to functional programming: Combinatorial linear logic, categorical and linear machine
- Game semantics